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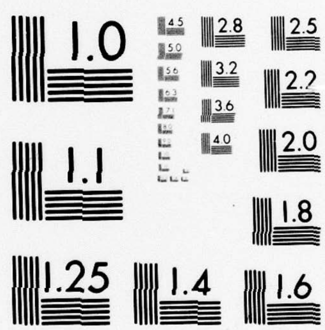

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ANALYSIS OF MIXED METHODS USING MESH DEPENDENT NORMS

I. Babuška, J. Osborn and J. Pitkäranta

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ANALYSIS OF MIXED METHODS USING MESH DEPENDENT NORMS

I. Babuška[†], J. Osborn[‡], and J. Pitkäranta[‡‡]

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ABSTRACT

This paper analyzes mixed methods for the biharmonic problem by means of new families of mesh dependent norms which are introduced and studied. More specifically, several mixed methods are shown to be stable with respect to these norms and, as a consequence, error estimates are obtained in a simple and direct manner.

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SIGNIFICANCE AND EXPLANATION

This paper presents a new approach to the analysis of mixed methods for the approximate solution of 4th order elliptic boundary value problems. In this approach one introduces a pair of mesh dependent norms and proves the approximation method is stable with respect to these norms. The error estimates then follow in a direct manner. In a mixed method, one introduces an auxiliary variable, usually representing another physically important quantity, and writes the differential equation as a lower order system. One then considers Ritz-Galerkin approximation schemes based on a variational formulation of this lower order system, thereby obtaining direct approximations to both the original and auxiliary variables. Three particular mixed methods for the approximate solution of the biharmonic problem are examined in detail.

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ANALYSIS OF MIXED METHODS USING
MESH DEPENDENT NORMS

I. Babuška [†], J. Osborn [‡], and J. Pitkäranta [‡‡]

1. Introduction

In [5] Brezzi studied Ritz-Galerkin approximation of saddle point problems arising in connection with Lagrange multipliers. These problems have the form:

Given $f \in \mathcal{V}'$ and $g \in \mathcal{W}'$, find $(u,\psi) \in \mathcal{V} \times \mathcal{W}$ satisfying

$$(1.1)\quad \begin{cases} a(u,v) + b(v,\psi) = (f,v) \ \forall\, v \in \mathcal{V} \\ \qquad\qquad b(u,\varphi) = (g,\varphi) \ \forall\, \varphi \in \mathcal{W}, \end{cases}$$

where $\mathcal{V}$ and $\mathcal{W}$ are real Hilbert spaces and $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are bounded bilinear forms on $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{W}$, respectively.

Given finite dimensional spaces $V_h \subset \mathcal{V}$ and $W_h \subset \mathcal{W}$, indexed by the parameter $0 < h < 1$, the Ritz-Galerkin approximation (u_h,ψ_h) to (u,ψ) is defined as the solution of the problem:

Find $(u_h,\psi_h) \in V_h \times W_h$ satisfying

$$(1.2)\quad \begin{cases} a(u_h, v) + b(v,\psi_h) = (f,v) \quad \forall \quad v \in V_h \\ \qquad\qquad b(u_h,\varphi) = (g,\varphi) \quad \forall \quad \varphi \in W_h. \end{cases}$$

The major assumptions in Brezzi's results are

$$(1.3)\qquad \sup_{v \in Z_h} \frac{|a(u,v)|}{\|v\|_{\mathcal{V}}} \ge \gamma_0 \|u\|_{\mathcal{V}} \quad \forall \ u \in Z_h \quad \text{and} \quad \forall\, h,$$

where $\gamma_0 > 0$ is independent of h, and $Z_h = \{v \in V_h : b(v,\varphi) = 0 \ \ \forall\, \varphi \in W_h\}$, and

$$(1.4)\qquad \sup_{v \in V_h} \frac{|b(v,\varphi)|}{\|v\|_{\mathcal{V}}} \ge k_0 \|\varphi\|_{\mathcal{W}} \quad \forall \ \varphi \in W_h \quad \text{and} \quad \forall\, h,$$

where $k_0 > 0$ is independent of h. Using (1.3) and (1.4) Brezzi proves the following error estimate for the approximation method determined by (1.2):

$$(1.5)\qquad \|u-u_h\|_{\mathcal{V}} + \|\psi-\psi_h\|_{\mathcal{W}} \le C(\inf_{\chi \in V_h}\|u-\chi\|_{\mathcal{V}} + \inf_{\eta \in W_h}\|\psi-\eta\|_{\mathcal{W}}) \ \forall\, h,$$

where C is independent of h.

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In [1,2] Babuška studied Ritz-Galerkin approximation of general, variationally posed problems. The main result of [1,2], as applied to (1.1) and (1.2), is that (1.5) holds provided

$$(1.6) \qquad \sup_{(v,\varphi)\in V_h\times W_h} \frac{|a(u,v) + b(v,\psi) + b(\eta,\varphi)|}{\|v\|_V + \|\varphi\|_W} \geq \tau_0(\|u\|_V + \|\psi\|_W) \ \forall (u,\psi) \in V_h \times W_h \text{ and } \forall\, h,$$

where $\tau_0 > 0$ is independent of h. It is clear from [1,2] that (1.3) and (1.4) hold if and only if (1.6) holds. (1.3)-(1.4) or, equivalently, (1.6) is referred to as the stability condition for this approximation method.

The results of [1,2,5] can be viewed as a stratagy for analyzing such approximation methods: the approximation method is characterized by certain bilinear forms, norms (spaces), and families of finite dimensional approximation spaces, and if the method can be shown to be stable with respect to the chosen norms, then the error estimates in these norms follow directly, provided the bilinear forms are bounded and the approximation properties of V_h and W_h are known in these norms. These results can be used to analyze, for example, certain hybrid methods for the biharmonic problem [5,6]. The results of [1,2] have also been used to analyze a variety of variationally posed problems that do not have the form (1.1).

There are other problems of a similar nature, however, where attempts at using the results of [1,2,5] were not entirely successful since not all of the hypotheses were satisfied: specifically, the Brezzi condition (1.3) or, equivalently, the Babuška condition (1.6), is not satisfied with the usual choice of norms, i.e., the approximation methods for these problems are not stable with respect to the usual choice of norms. This is the case, for example, in the analysis [7] of the Herrmann-Miyoshi [15,16,20] mixed method for the biharmonic problem. In the analysis of this method a natural choice for both $\|\cdot\|_V$ and $\|\cdot\|_W$ is the 1st order Sobolev norm; however, this method is not stable with respect to this choice of norms. As a result of this difficulty the error estimates obtained in [5] are not optimal. A similar difficulty arises in the analysis of the Herrmann-Johnson [15, 16,17] and Ciarlet-Raviart [9] mixed methods for the biharmonic problem. In later work of Scholz [23] and Rannacher [22] optimal error estimates were obtained for the mixed methods of Ciarlet-Raviart and Herrmann-Miyoshi. In a forthcoming paper Falk-Osborn [12] develop abstract results from which optimal error estimates for these and other problems can be derived. However, in neither the work of Scholz [23], Rannacher [22], or Falk-Osborn [12] is the systematic approach of Brezzi and Babuška used.

It is the purpose of this paper to analyze mixed methods for the biharmonic problem by means of the results of Brezzi and Babuška. This is done by introducing a new family of (mesh dependent) norms with respect to which the above mentioned mixed methods (Ciarlet-Raviart, Herrmann-Miyoshi, Herrmann-Johnson) are stable. Once the stability condition has been checked and the approximation properties of the subspaces V_h and W_h have been determined in these new norms, the error estimates in these norms follow immediately from the abstract results of Brezzi and Babuška. Error estimates in the more standard norms are then obtained by using the usual duality argument. The results of this paper were announced in [21]. We also note that the methods employed in this paper have been applied to two point boundary value problems in [3].

Section 2 contains a review of the convergence results of Brezzi and Babuška. In Section 3 we introduce and study the mesh dependent norms and spaces used in the analysis in this paper. In Section 4 we treat three examples previously analyzed in the literature and show how error estimates can be derived from the abstract results in Section 2, used in conjunction with the mesh dependent norms introduced in Section 3. These examples are all mixed methods for the biharmonic problem. The error estimates in the standard norms that are obtained in the present paper and those obtained in [12], using different techniques, are the same.

Throughout this paper we will use the Sobolev spaces $H^m = H^m(\Omega)$, where Ω is a convex polygon in the plane and m is a nonnegative integer. On these spaces we have the seminorms and norms

$$|v|_m = |v|_{m,\Omega} = \Big(\sum_{|\alpha|=m} \int_\Omega |D^\alpha v|^2 dx\Big)^{1/2}$$

and

$$\|v\|_m = \|v\|_{m,\Omega} = \Big(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha v|^2 dx\Big)^{1/2} .$$

$H_0^m(\Omega)$ denotes the subspace of $H^m(\Omega)$ of functions vanishing together with their first $m-1$ normal derivatives on $\Gamma = \partial\Omega$. We also use the spaces $H^{-m}(\Omega) = (H_0^m(\Omega))'$ (the dual space of $H_0^m(\Omega)$) with the norm on $H^{-m}(\Omega)$ taken to be the usual dual norm.

2. Abstract Convergence Results

In this section we review certain results on the approximate solution of saddle point problems.

Let $\mathcal{V}_h$ and $\mathcal{W}_h$ be real Hilbert spaces (indexed by the parameter h, where $0 < h < 1$) with norms $\|\cdot\|_{\mathcal{V}_h}$ and $\|\cdot\|_{\mathcal{W}_h}$, respectively, and let $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ be bilinear forms on $\mathcal{V}_h \times \mathcal{V}_h$ and $\mathcal{V}_h \times \mathcal{W}_h$, respectively. We suppose

(2.1) $$|a_h(u,v)| \le K_1 \|u\|_{\mathcal{V}_h} \|v\|_{\mathcal{V}_h} \quad \forall \quad u,v \in \mathcal{V}_h ,$$

(2.2) $$|b_h(u,\varphi)| \le K_2 \|u\|_{\mathcal{V}_h} \|\varphi\|_{\mathcal{W}_h} \quad \forall \quad u \in \mathcal{V}_h, \ \forall \ \varphi \in \mathcal{W}_h ,$$

where K_1 and K_2 are constants that do not depend on h.

We consider the following problem, referred to as problem P:

Given $f \in \mathcal{V}_h'$ and $g \in \mathcal{W}_h'$, find $(u,\psi) \in \mathcal{V}_h \times \mathcal{W}_h$ satisfying

(2.3a) $$a_h(u,v) + b_h(v,\psi) = (f,v) \quad \forall \quad v \in \mathcal{V}_h ,$$

(2.3b) $$b_h(u,\varphi) = (g,\varphi) \quad \forall \quad \varphi \in \mathcal{W}_h ,$$

where $(\cdot,\cdot)$ denotes the pairing between $\mathcal{V}_h$ and its dual space $\mathcal{V}_h'$, or between $\mathcal{W}_h$ and $\mathcal{W}_h'$.

We shall consider this problem for a subclass of data, i.e., for $(f,g) \in D$, where D is a subclass of $\mathcal{V}_h' \times \mathcal{W}_h'$. We assume that P has a unique solution for all $(f,g) \in D$.

We are interested in the approximate solution of P. Toward this end we suppose we are given finite dimensional spaces $V_h \subset \mathcal{V}_h$ and $W_h \subset \mathcal{W}_h$, $0 < h < 1$, and consider the following problem, referred to as problem P_h:

Given $(f,g) \in D$, find $(u_h,\psi_h) \in V_h \times W_h$ satisfying

(2.4a) $$a_h(u_h,v) + b_h(v,\psi_h) = (f,v) \quad \forall \quad v \in V_h ,$$

(2.4b) $$b_h(u_h,\varphi) = (g,\varphi) \quad \forall \quad \varphi \in W_h .$$

We now regard u_h as an approximation to u and ψ_h as an approximation to ψ.

Regarding problem P_h we suppose

(2.5) $$\sup_{v \in Z_h} \frac{|a_h(u,v)|}{\|v\|_{\mathcal{V}_h}} \ge \gamma_0 \|u\|_{\mathcal{V}_h} \quad \forall \quad u \in Z_h \quad \text{and} \quad \forall h ,$$

where $\gamma_0 > 0$ is independent of h and $Z_h \equiv \{v \in V_h : b_h(v,\varphi) = 0 \quad \forall \quad \varphi \in W_h\}$, and

(2.6) $$\sup_{v \in V_h} \frac{|b_h(v,\varphi)|}{\|v\|_{\mathcal{V}_h}} \geq k_0 \|\varphi\|_{\mathcal{W}_h} \quad \forall \ \varphi \in W_h \ \text{ and } \ \forall\, h ,$$

where $k_0 > 0$ is independent of h. We now state the fundamental estimate for the errors $u - u_h$ and $\psi - \psi_h$.

Theorem 1 (Brezzi [5]). Suppose (2.1), (2.2), (2.5) and (2.6) are satisfied. Then Problem P_h has a unique solution (u_h, ψ_h) for each h and there is a constant C, independent of h, such that

(2.7) $$\|u-u_h\|_{\mathcal{V}_h} + \|\psi-\psi_h\|_{\mathcal{W}_h} \leq C\Big(\inf_{\chi \in V_h} \|u-\chi\|_{\mathcal{V}_h} + \inf_{\eta \in \mathcal{W}_h} \|\psi-\eta\|_{\mathcal{W}_h} \Big) \quad \forall\, h .$$

(2.5)-(2.6) is referred to as the stability condition for this approximation method.

In many applications of Theorem 1 the spaces $\mathcal{V}_h$ and $\mathcal{W}_h$ and the forms a_h and b_h do not depend on h, i.e., $\mathcal{V}_h = \mathcal{V}$ and $\mathcal{W}_h = \mathcal{W}$ are fixed Hilbert spaces and $a_h = a$ and $b_h = b$ are fixed bilinear forms and $\mathcal{V} \times \mathcal{V}$ and $\mathcal{V} \times \mathcal{W}$. The space V_h and W_h typically are spaces of piecewise polynomials with respect to a triangulation T_h of some domain by triangles of size less than or equal to h and, of course, depend on h. In the applications in this paper, both the spaces $\mathcal{V}_h$, $\mathcal{W}_h$ and V_h, W_h depend on h, i.e., are mesh dependent; the constants K_1, K_2, γ_0, and k_0, however, will be independent of h (cf. [2, Cp. 7]). In these applications the solution (u,ψ) of (2.3) is independent of h and lies in $\mathcal{V}_h \times \mathcal{W}_h$ for all h. Thus the estimate (2.7) yields convergence estimates for $u-u_h$ and $\psi-\psi_h$, provided the families $\{V_h\}$ and $\{W_h\}$ satisfy an approximobility assumption. For typical finite element applications this would involve the assumption that $\inf_{\chi \in V_h} \|u-\chi\|_{\mathcal{V}_h}$ and $\inf_{\eta \in W_h} \|\psi-\eta\|_{\mathcal{W}_h}$ tend to zero as h tends to zero.

3. Mesh Dependent Norms and Spaces

In this section we describe the mesh dependent norms and spaces we shall use in the paper. Let Ω be a convex polygon in the plane. For $0 < h < 1$ we let T_h be a triangulation of Ω by triangles T of diameter less than or equal to h. We assume the family of triangulations $\{T_h\}$ satisfies the minimal angle condition, i.e., there is a constant σ such that

$$(3.1) \qquad \max_{T \in T_h} \frac{h_T}{\rho_T} \le \sigma \qquad \forall\, h,$$

where h_T is the diameter of T and ρ_T is the diameter of the largest circle contained in T, and is quasi-uniform, i.e., there is a constant $\tau > 0$ such that

$$(3.2) \qquad \frac{h}{h_T} \le \tau \quad \forall\, T \in T_h \text{ and } \forall\, h.$$

Let $\Gamma_h = \bigcup_{T \in T_h} \partial T$. We define

$$H_h^2 = \{u \in H^1(\Omega) : u|_T \in H^2(T)\ \forall\, T \in T_h\}$$

and on H_h^2 define the norm

$$\|u\|_{2,h}^2 = \sum_{T \in T_h} \|u\|_{2,T}^2 + h^{-1} \int_{\Gamma_h} |J \frac{\partial u}{\partial \nu}|^2 ds,$$

where, if $T' = \partial T^1 \cap \partial T^2$ is an interior edge of the triangulation T_h, we set $J \frac{\partial u}{\partial \nu}\Big|_{T'} = \frac{\partial u}{\partial \nu^1} + \frac{\partial u}{\partial \nu^2}$, where ν^j is the unit normal to T' exterior to T^j, and if T' is a boundary edge of T_h, we set $J \frac{\partial u}{\partial \nu}\Big|_{T'} = \frac{\partial u}{\partial \nu}$.

On $H^1(\Omega)$ we define

$$\|u\|_{0,h}^2 = \int_\Omega |u|^2 dx + h \int_{\Gamma_h} |u|^2 ds$$

and then define H_h^0 to be the completion of $H^1(\Omega)$ with respect to $\|\cdot\|_{0,h}$. H_h^0 can be identified with $L_2(\Omega) \oplus L_2(\Gamma_h)$.

We note that norms similar to $\|\cdot\|_{0,h}$ and $\|\cdot\|_{2,h}$ have been used in a different manner in Douglas-Dupont [11] and Thomas [26].

For $k \ge 1$ a fixed integer we define

$$(3.3) \qquad S_h = \{v \in C^0(\bar{\Omega}) : v|_T \in P_k \quad \forall \quad T \in T_h\}$$

where P_k is the space of polynomials of degree k or less in the variable x_1 and x_2. It is clear that S_h is contained in H_h^0 and H_h^2.

We now prove several lemmas that are fundamental to the analysis of this paper. These proofs are all closely related to the ideas used in the proof of the Bramble-Hilbert lemma [4]. Prior to stating the first of these lemmas we describe the notation we will use and state some well-known results that will be used in the proofs.

Let T be an arbitrary triangle and let $\hat{T}$ be the reference triangle with vertices (0,0), (1,0), and (0,1). Then there is an invertible affine mapping $F_T(\hat{x}) = B_T\hat{x} + b_T = F(\hat{x}) = B\hat{x} + b$ such that $T = F_T(\hat{T})$. This mapping leads to the correspondence $\hat{x} \in \hat{T} \leftrightarrow x = F_T(\hat{x}) \in T$ between points in $\hat{T}$ and points in T and the correspondence $(\hat{v}:\hat{T} \to R) \leftrightarrow (v = \hat{v} \circ F_T^{-1}:T \to R)$ between functions defined on $\hat{T}$ and functions defined on T. Note that $\hat{v}(\hat{x}) = v(x)$.

It is easily seen that

$$(3.4) \qquad (\nabla_x v)(x) = (B^{-1})^t (\nabla_{\hat{x}}\hat{v})(F^{-1}(x)).$$

If $\nu = \nu(x)$ denotes the outward unit normal to ∂T at x and $\hat{\nu} = \hat{\nu}(\hat{x})$ is the outward unit normal to $\partial\hat{T}$ and $\hat{x}$, then

$$(3.5) \qquad \nu(x) = (B^{-1})^t\, \hat{\nu}(\hat{x})\, |B^t\, \nu(x)|$$

where t denotes transpose. Let the sides of T be denoted by T_i', $i=1,2,3$. $|T|$ denotes the area of T and $|T_i'|$ denotes the length of T_i'. The seminorms $|v|_{\ell,T}$ and $|\hat{v}|_{\ell,\hat{T}}$ are related by

$$(3.6) \qquad |\hat{v}|_{\ell,\hat{T}} \le |\det B|^{-1/2}\, \|B\|^{\ell} |v|_{\ell,T}$$

and

$$(3.7) \qquad |v|_{\ell,T} \le |\det B|^{1/2}\, \|B^{-1}\|^{\ell}\, |\hat{v}|_{\ell,\hat{T}}$$

where $\|B\|$ is the norm of B induced by the Euclidian vector norm (cf. [8, Theorem 3.1.2]). We will also use the estimates

$$(3.8) \qquad \|B\| \le \frac{h_T}{\rho_{\hat{T}}}\ , \quad \|B^{-1}\| \le \frac{h_{\hat{T}}}{\rho_T}$$

(cf. [8, Theorem 3.1.3]). We also note that $|\det B| = \frac{|T|}{|\hat{T}|}$. Finally we remark that there is a constant $C = C(\hat{T})$ such that

$$(3.9) \qquad \inf_{p\in P_k} \|\hat{u} + p\|_{k+1,\hat{T}} \le C|\hat{u}|_{k+1,\hat{T}} \qquad \forall\ \hat{u} \in H^{k+1}(\hat{T})$$

(cf. [8, Theorem 3.1.1]).

Lemma 1. There is a constant C such that

$$\|u\|_{0,h} \le C\|u\|_0 \qquad \forall \quad u \in S_h .$$

Proof. It is sufficient to show that

$$h \int_{\Gamma_h} |u|^2 ds \le C\|u\|_0^2 \qquad \forall \quad u \in S_h .$$

Now $(\int_{\hat{T}} |\hat{u}|^2 dx)^{1/2}$ and $(\int_{\hat{T}} |\hat{u}|^2 dx + \int_{\partial\hat{T}} |\hat{u}|^2 ds)^{1/2}$ are both norms on the finite dimensional space $P_k(\hat{T}) = \{p|_{\hat{T}} : p \in P_k\}$ and hence there is a constant $C(\hat{T})$ such that

$$\int_{\partial\hat{T}} |\hat{u}|^2 ds \le C(\hat{T}) \int_{\hat{T}} |\hat{u}| d\hat{x} \qquad \forall\, \hat{u} \in P_k(\hat{T}) .$$

Let $T \in \mathcal{T}_h$ and suppose T is the image of $\hat{T}$ under the mapping $F(\hat{x}) = B\hat{x} + b$. Then, using (3.1), (3.2), and (3.6), we see that for any $u \in P_k$ we have

$$\begin{aligned}
\int_{\partial T} |u|^2 ds &= \sum_{i=1}^3 \int_{T_i'} |u|^2 ds \\
&\le \sum_i \int_{\hat{T}_i'} |\hat{u}|^2 |T_i'| d\hat{s} \\
&\le C(\hat{T}) \max_i |T_i'| \int_{\hat{T}} |\hat{u}|^2 d\hat{x} \\
&\le C(\hat{T}) \max_i |T_i'| |\det B|^{-1} \int_T |u|^2 dx \\
&\le C(\hat{T}) \max_i |T_i'| \frac{|\hat{T}|}{|T|} \|u\|_{0,T}^2 \\
&\le \frac{C(\hat{T})\ 4|\hat{T}|}{\pi} \frac{h_T}{\rho_T{}^2} \|u\|_{0,T}^2 \\
&\le \frac{C(\hat{T})\ 4|\hat{T}|}{\pi} \left(\frac{h_T}{\rho_T}\right)^2 \frac{1}{h_T} \|u\|_{0,T}^2 \\
&\le C(\hat{T})\, \sigma h_T^{-1} \|u\|_{0,T}^2 \\
&\le C(\hat{T})\, \sigma\, \tau\, h^{-1} \|u\|_{0,T}^2 .
\end{aligned}$$

Therefore

$$h \int_{\Gamma_h} |u|^2 ds \le h \sum_{T \in \mathcal{T}_h} \int_{\partial T} |u|^2 ds$$

$$\le C(\hat{T})\ \sigma\ \tau \sum_{T \in \mathcal{T}_h} \|u\|^2_{0,T}$$

$$\le C(\hat{T})\ \sigma\ \tau\ \|u\|^2_{0,\Omega}$$

for all $u \in S_h$.

Lemma 2. There is a constant C such that

$$\|u\|_{2,h} \le C\ h^{-1} \|u\|_{1,\Omega} \quad \forall \quad u \in P_k .$$

Proof. Since $\{\mathcal{T}_h\}$ is quasi-uniform it is well-known that

$$\sum_{T \in \mathcal{T}_h} \|u\|^2_{2,T} \le C\ h^{-2}\ \|u\|^2_{1,\Omega} \qquad \forall \quad u \in P_k .$$

Thus it is sufficient to show that

$$h^{-1} \int_{\Gamma_h} |J \frac{\partial u}{\partial \nu}|^2\ ds \le C\ h^{-2}\ \|u\|^2_{1,\Omega} \qquad \forall \quad u \in P_k .$$

$(\int_{\hat{T}} |\hat{u}|^2 dx + \int_{\partial \hat{T}} |\nabla_{\hat{x}} \hat{u}|^2 d\hat{s})^{1/2}$ and $\|\hat{u}\|_{1,\hat{T}}$ are both norms on the finite dimensional space $P_k(\hat{T})$ and hence there is a constant $C(\hat{T})$ such that

$$E(\hat{u}) \equiv \int_{\partial \hat{T}} |\nabla_{\hat{x}} \hat{u}|^2 d\hat{s} \le C(\hat{T}) \|\hat{u}\|^2_{1,\hat{T}} \qquad \forall \quad \hat{u} \in P_k(\hat{T}) .$$

Clearly $\quad E(\hat{u} + p) = E(\hat{u}) \quad \forall \quad p \in P_0 .$ Thus

$$E(\hat{u}) = E(\hat{u} + p) \le C(\hat{T})\ \|\hat{u} + p\|^2_{1,\hat{T}} \qquad \forall \quad p \in P_0$$

and hence, using (3.9), we have

$$E(\hat{u}) \le C(\hat{T}) \inf_{p \in P_0} \|\hat{u} + p\|_{1,\hat{T}}$$

$$\le C(\hat{T}) |\hat{u}|_{1,\hat{T}} .$$

Now let $T \in \mathcal{T}_h$ and assume T is the image of $\hat{T}$ under the mapping $F(x) = Bx + b$. Then, using (3.1), (3.2), (3.4), (3.6), and (3.8), we see that for any $u \in P_k$ we have

$$\int_{\partial T} \left|\frac{\partial}{\partial \nu}\right|^2 ds = \sum_i \int_{T_i'} \left|[(\nabla_x u)(x)]^t \nu(x)\right|^2 ds$$

$$\leq \sum_i \int_{T_i'} \left|(B^{-1})^t (\nabla_{\hat{x}} \hat{u})(F^{-1}(x))\right|^2 ds$$

$$\leq |B^{-1}|^2 \max_i |T_i'| \int_{\partial \hat{T}} |\nabla_{\hat{x}} \hat{u}|^2 d\hat{s}$$

$$\leq C(\hat{T}) \|B^{-1}\|^2 \max |T_i'| |\hat{u}|^2_{1,\hat{T}}$$

$$\leq C(\hat{T}) \|B^{-1}\|^2 \max |T_i'| |\det B|^{-1} \|B\|^2 |u|^2_{1,T}$$

$$\leq C(\hat{T}) \left(\frac{h_{\hat{T}}}{\rho_T}\right)^2 \left(\frac{h_T}{\rho_{\hat{T}}}\right)^2 h_T \frac{|\hat{T}|}{|T|} |u|^2_{1,T}$$

$$\leq C(\hat{T}) \left(\frac{h_{\hat{T}}}{\rho_{\hat{T}}}\right)^2 \left(\frac{h_T}{\rho_T}\right)^4 \frac{4|\hat{T}|}{\pi} \frac{1}{h_T} |u|^2_{1,T}$$

$$\leq C(\hat{T}) \left(\frac{h_{\hat{T}}}{\rho_{\hat{T}}}\right)^2 \sigma^4 \tau \frac{4|\hat{T}|}{\pi} h^{-1} |u|^2_{1,T} .$$

Therefore we obtain

$$h^{-1} \int_{\Gamma_h} \left|J\frac{\partial u}{\partial h}\right|^2 ds \leq \sum_{T \in T_h} h^{-1} \int_{\partial T} \left|\frac{\partial u}{\partial \nu}\right|^2 ds$$

$$\leq C(\hat{T}) \sigma^4 \tau \sum_{T \in T_h} h^{-2} |u|^2_{1,T}$$

$$\leq C(\hat{T}) \sigma^4 \tau h^{-2} |u|^2_{1,\Omega} .$$

This completes the proof.

Lemma 3. There is a constant C such that

$$\inf_{\chi\in S_h} \|u-\chi\|_{0,h} \leq C\, h^{\ell}|u|_{\ell,\Omega}$$

for all $u \in H^r(\Omega)$ and all h, where $1 \leq r$ and $1 \leq \ell \leq \min(r,k+1)$.

Proof. We define two interpolation operators that will be used in the proof. For $u \in H^2(T)$ let $I_T u \in P_k$ be defined by

$$\int_T (u - I_T u) f\, dx = 0 \qquad \forall\, f \in P_{k-3}\,,$$

$$\int_{T'} (u - I_T u) f\, dx = 0 \qquad \forall\, f \in P_{k-2} \text{ and } \forall \text{ sides } T' \text{ of } T\,,$$

and

$$u(a) - (I_T u)(a) = 0 \qquad \forall \text{ vertices } a \text{ of } T\,.$$

Then for $u \in H^2(\Omega)$ we let $I_h u \in S_h$ be defined by

$$(I_h u)\big|_T = I_T (u\big|_T)\,.$$

For $u \in H^1(\Omega)$ we define the interpolant in a different manner. Here we consider only the case $k = 1$. Let the vertices of T_h be denoted by $z_1,\ldots,z_m$ and let $w_1,\ldots,w_m$ be the basis for S_h defined by $w_i(z_j) = \delta_{ij}$. Set $S_j = (\operatorname{supp} w_j) \cap \Omega$ and let $|S_j|$ be the area of S_j. Now, following Clément [10] we define $\tilde{I}_h u$ by

$$\tilde{I}_h u = \sum_{j=1}^{m} \frac{\int_{S_j} u\, dx}{|S_j|}\, w_j\,.$$

We first consider the case $r \geq 2$ and $\ell \geq 2$. In this case we obtain the desired result by estimating $\|u - I_h u\|_{0,h}$. By the standard approximability results for S_h we have

$$\int_\Omega |u - I_h u|^2 dx \leq C\, h^{2\ell}|u|^2_{\ell,\Omega}\,.$$

Thus it is sufficient to show that

$$\int_{\Gamma_h} |u - I_h u|^2 ds \leq C\, h^{2\ell-1}|u|^2_{\ell,\Omega}\,.$$

Suppose $u \in H^{\ell}(\hat{T})$ and set $E(\hat{u}) = \int_{\partial T} |\hat{u} - I_{\hat{T}}\hat{u}|^2 d\hat{s}$. By the trace theorem and the Sobolev imbedding theorem we have

$$E(\hat{u}) \le C(\hat{T}) \|\hat{u}\|^2_{\ell,\hat{T}}$$

and since $E(\hat{u} + p) = E(\hat{u})\ \forall\ p \in P_{\ell-1}$, we thus have

$$E(\hat{u}) \le C(\hat{T}) \inf_{p \in P_{\ell-1}} \|\hat{u} + p\|^2_{\ell,\hat{T}} \le C(\hat{T})\ |u|^2_{\ell,\hat{T}}\ .$$

Now let $T \in \mathcal{T}_h$ be the image of $\hat{T}$ under the mapping $F(\hat{x}) = B\hat{x} + b$.

Then

$$\begin{aligned}
\int_{\partial T} |u - I_T u|^2 ds &\le \sum_i \int_{\hat{T}_i'} |\hat{u} - I_{\hat{T}}\hat{u}|^2 |T_i'|\, d\hat{s} \\
&\le \max_i |T_i'| \int_{\partial\hat{T}} |\hat{u} - I_{\hat{T}}\hat{u}|^2 d\hat{s} \\
&\le \max_i |T_i'|\, C(\hat{T})\, |\hat{u}|^2_{\ell,\hat{T}} \\
&\le \max_i |T_i'| C(\hat{T}) |\det B|^{-1} \|B\|^{2\ell} |u|^2_{\ell,T} \\
&\le C(\hat{T}) |\hat{T}| \frac{4h_T}{\pi \rho_T^2} \left[\frac{h_T}{\rho_{\hat{T}}} \right]^{2\ell} |u|^2_{\ell,T} \\
&\le \frac{C(\hat{T}) |\hat{T}|\ 4\ \sigma^2}{\pi\ \rho_{\hat{T}}^{2\ell}}\ h_T^{2\ell-1}\ |u|^2_{\ell,T}\ .
\end{aligned}$$

Therefore

$$\int_{\Gamma_h} |u - I_h u|^2 ds \le \sum_{T\in T_h} \int_{\partial T} |u - I_T u|^2 \, ds$$

$$\le C(\hat{T}) \, \sigma^2 \, h^{2\ell-1} \sum_{T\in T_h} |u|^2_{\ell,T}$$

$$= C(\hat{T}) \, \sigma^2 \, h^{2\ell-1} \, |u|^2_{\ell,\Omega} .$$

This completes the proof for the case $r,\ell \ge 2$.

For the case $r \ge 2$ and $\ell = 1$ or $r = \ell = 1$ we estimate $\|u - \tilde{I}_h u\|_{0,h}$. Clément [10] has shown that

$$\|u - \tilde{I}_h u\|_0 \le C \, h|u|_1 .$$

By a slight modification of the proof in [9] we obtain

$$(h \int_{\Gamma_h} |u - \hat{I}_h u|^2 ds)^{1/2} \le C \, h|u|_1 .$$

The desired result now follows.

Lemma 4. There is a constant C such that

$$\inf_{\chi\in S_h\cap H_0^1} \|u - \chi\|_{2,h} \le C \, h^{\ell-2} \, |u|_{\ell,\Omega}$$

for all $u \in H^r(\Omega) \cap H_0^1(\Omega)$ and all h , where $2 \le r$ and $2 \le \ell \le \min(r,k+1)$.

Proof. Let I_h be defined as in the proof of Lemma 3 . Note that $I_h u \in S_h \cap H_0^1$ if $u \in H^r \cap H_0^1$. Since, by standard approximability results we have

$$\sum_T \|u - I_h u\|^2_{2,T} \le C \, h^{2\ell-4} \, |u|^2_{\ell,\Omega},$$

it is sufficient to show that

$$\int_{\Gamma_h} |J \frac{\partial(u - I_h u)}{\partial\nu}|^2 ds \le C h^{2\ell-3} \, |u|^2_{\ell,\Omega} .$$

We next observe that

$$\int_{\partial\hat{T}} |\nabla_{\hat{x}}(u - I_{\hat{T}}\hat{u})|^2 ds \le C(\hat{T}) \, |\hat{u}|^2_{\ell,T} \qquad \forall \, \hat{u} \in H^\ell(\hat{T}) .$$

Now let $T \in T_h$ be the image of $\hat{T}$ under the mapping $F(\hat{x}) = B\hat{x} + b$.

Then

$$\int_{\partial} \left|\frac{\partial}{\partial\nu}(u - I_T u)\right|^2 ds = \sum_i \int_{T_i'} |[\nabla_x (u - I_T u)]^t \nu(x)|^2 ds$$

$$= \sum_i \int_{T_i'} |(B^{-1})^t \nabla_{\hat{x}}(\hat{u} - I_{\hat{T}}\hat{u})|^2 ds$$

$$\leq \sum_i \int_{\hat{T}_i'} |(B^{-1})^t \nabla_{\hat{x}}(\hat{u} - I_{\hat{T}}\hat{u})(\hat{x})|^2 |T_i'| d\hat{s}$$

$$\leq |B^{-1}|^2 \max_i |T_i'| \int_{\partial\hat{T}} |\nabla_{\hat{x}}(\hat{u} - I_{\hat{T}}\hat{u})|^2 d\hat{s}$$

$$\leq C(\hat{T}) \|B^{-1}\|^2 \max|T_i'| |\hat{u}|^2_{\ell,\hat{T}}$$

$$\leq C(\hat{T}) \left(\frac{h_{\hat{T}}}{\rho_T}\right)^2 \left(\frac{h_T}{\rho_{\hat{T}}}\right)^{2\ell} \frac{|\hat{T}|}{|T|} |u|^2_{\ell,T}\, h_T$$

$$\leq \frac{C(\hat{T})|\hat{T}|\, 4\, h_{\hat{T}}^2}{\pi\, \rho_{\hat{T}}^{2\ell}} \sigma^4 h_T^{2\ell-3} |u|^2_{\ell,T} .$$

Therefore

$$\int_{\Gamma_h} \left|J \frac{\partial(u-u_T)}{\partial\nu}\right|^2 ds \leq \sum_{T\in T_h} \int_{\partial T} \left|\frac{\partial}{\partial\nu}(u - I_T u)\right|^2 ds$$

$$\leq C(\hat{T})\, \sigma^4 h^{2\ell-3} |u|^2_{\ell,\Omega} ,$$

which completes the proof.

4. Applications

In this section we analyze three mixed methods.

a) Ciarlet-Raviart method

Consider the biharmonic problem

$$(4.1)\qquad \begin{cases} \Delta^2\psi = g & \text{in } \Omega \\ \psi = \dfrac{\partial\psi}{\partial\nu} = 0 & \text{on } \Gamma = \partial\Omega \end{cases}$$

where Ω is a convex polygon in the plane and g is a given function. If $g \in H^{-2}(\Omega)$ then there is a unique solution $\psi \in H_0^2(\Omega)$ of (4.1). In addition the following regularity result is known for this problem: If $g \in H^{-1}(\Omega)$, then $\psi \in H^3(\Omega) \cap H_0^2(\Omega)$ and there is a constant C such that

$$(4.2)\qquad \|\psi\|_3 \le C\|g\|_{-1} \qquad \forall\ g \in H^{-1}(\Omega)\ .$$

Using the well-known correspondence between the biharmonic problem and the Stokes problem, this regularity result can be deduced from the regularity result for the Stokes problem proved in [18]. We assume $g \in H^{-1}(\Omega)$ throughout this section.

We now seek an approximation to the solution ψ of (4.1) by a mixed method, i.e., we introduce an auxiliary variable ($u \equiv -\Delta\psi$ for the method of this subsection), write (4.1) as a second order system, cast this system into variational form, and then consider the Ritz-Galerkin method corresponding to this variational formulation.

Thus we let $u \equiv -\Delta\psi$ and write (4.1) as

$$(4.3)\qquad \begin{cases} \Delta u = -g \\ \Delta\psi + u = 0 \quad \text{in } \Omega \\ \psi = \dfrac{\partial\psi}{\partial\nu} = 0 \text{ on } \Gamma\ . \end{cases}$$

The desired variational formulation of (4.3) is obtained by multiplying the $1^{\underline{st}}$ equation in (4.3) by $\varphi \in H_h^2 \cap H_0^1$, the $2^{\underline{nd}}$ equation by $v \in H_h^0$, integrating the resulting equations over Ω, and integrating the first one by parts over each $T \in \mathcal{T}_h$. By means of this process we arrive at the following problem:

Given $g \in H^{-1}(\Omega)$, find $(u,\psi) \in H_h^0 \times (H_h^2 \cap H_0^1)$ satisfying

$$(4.4)\quad \begin{cases} \int_\Omega u\, v\, dx - \sum_{T\in\mathcal{T}_h} \int_T v\,\Delta\psi\, dx - \int_{\Gamma_h} v\left(J\,\frac{\partial\psi}{\partial\nu}\right) ds = 0 & \forall\, v \in H_h^0 \\ \sum_{T\in\mathcal{T}_h} \int_T u\,\Delta\varphi\, dx - \int_{\Gamma_h} u\left(J\frac{\partial\varphi}{\partial\nu}\right) ds = -\int_\Omega g\,\varphi\, dx & \forall\, \varphi \in H_h^2 \cap H_0^1 . \end{cases}$$

Using the regularity result (4.2) one can easily show that if ψ is the solution of (4.1) and $u \equiv -\Gamma\psi$, then (u,ψ) is a solution of (4.4), and if (u,ψ) is a solution of (4.4), then ψ is the solution of (4.1) and $u = -\Delta\psi$. (4.4) is an example of problem P in Section 2 with $V_h = H_h^0$, $\|\cdot\|_{V_h} = \|\cdot\|_{0,h}$, $W_h = H_h^2 \cap H_0^1$, $\|\cdot\|_{W_h} = \|\cdot\|_{2,h}$, $a_h(u,v) = \int_\Omega u\, v\, dx$, and

$$b_h(u,\varphi) = \sum_{T\in\mathcal{T}_h} \int_T u\Delta\varphi\, dx - \int_{\Gamma_h} u\left(J\,\frac{\partial\varphi}{\partial\nu}\right) ds ,$$

(and with g replaced by $-g$). Here the subclass of data for which (4.4) is uniquely solvable is $D = 0 \times H^{-1}(\Omega)$.

As pointed out above, H_h^0 can be identifies with $L_2(\Omega) \oplus L_2(\Gamma_h)$. Under this identification, $H^1(\Omega)$ is considered a linear manifold in H_h^0 through the mapping

$$H^1(\Omega) \ni u \to (u, u|_{\Gamma_h}) \in L_2(\Omega) \oplus L_2(\Gamma_h) = H_h^0 .$$

Thus an element $u = (\tilde{u},\tilde{\tilde{u}}) \in L_2(\Omega) \oplus L_2(\Gamma_h)$ is considered to be in $H^1(\Omega)$ if $\tilde{u} \in H^1(\Omega)$ and $\tilde{u}|_{\Gamma_h} = \tilde{\tilde{u}}$. To be completely precise b_h should be defined by

$$b_h(u,\varphi) = \sum_{T\in\mathcal{T}_h} \int_T \tilde{u}\Delta\varphi\, dx - \int_{\Gamma_h} \tilde{\tilde{u}}\,\left(J\,\frac{\partial\varphi}{\partial\nu}\right) ds$$

for $u = (\tilde{u},\tilde{\tilde{u}}) \in H_h^0 = L_2(\Omega) \oplus L_2(\Gamma_h)$ and $\varphi \in H_h^2$. Note that

$$(4.5)\qquad b_h(u,\varphi) = -\int_\Omega \nabla u \cdot \nabla\varphi\, dx$$

for $u \in H^1(\Omega)$ and $\varphi \in H_h^2$. We further note that it is immediate that (2.1) and (2.2) are satisfied with constants that do not depend on h.

For finite dimensional spaces we choose $V_h = S_h$ and $W_h = S_h \cap H_0^1(\Omega)$, where S_h is defined in (3.3). Problem P_h thus has the form:

Given $g \in H^{-1}(\Omega)$, find $(u_h, \psi_h) \in V_h \times W_h$ satisfying

$$(4.6) \quad \begin{cases} \int_\Omega u_h v\,dx + \sum_{T\in\mathcal{T}_h} \int_T v\,\Delta\psi_h\,dx - \int_{\Gamma_h'} v(J\frac{\partial\psi_h}{\partial\nu})dx = 0 & \forall\ v \in V_h \\ \sum_{T\in\mathcal{T}_h} \int_T u_h \Delta\varphi\,dx - \int_{\Gamma_h} u_n(J\frac{\partial\varphi}{\partial\nu})ds = -\int_\Omega g\varphi\,dx & \forall\ \varphi \in W_h . \end{cases}$$

Using (4.5) one easily sees that the approximation procedure determined by (4.6) is the same as that considered by Glowinski [14] and Mercier [19] and further developed by Ciarlet-Raviart [9]. Note that this method yields direct approximations to ψ and to $u = -\Delta\psi$ (the stream function and vorticity in hydrodunamical problems).

We have already observed that (2.1) and (2.2) are satisfied. In order to apply Theorem 1 we must check the stability condition (2.5)-(2.6).

Theorem 2. There is a constant $\gamma_0 > 0$, independent of h, such that

$$\sup_{v\in Z_n} \frac{|\int_\Omega u\,v\,dx|}{\|v\|_{0,h}} \geq \gamma_0 \|u\|_{0,h} \qquad \forall\quad u \in Z_h ,$$

i.e., (2.5) is satisfied.

Proof. Using Lemma 1 we have

$$\sup_{v\in Z_n} \frac{|\int_\Omega u\,v\,dx|}{\|v\|_{0,h}} \geq \sup_{v\in Z_h} \frac{|\int_\Omega u\,v\,dx|}{C\|v\|_0}$$

$$= C^{-1}\|u\|_0$$

$$\geq C^{-2}\|u\|_{0,h} \qquad \forall\quad u \in Z_h ,$$

where C is independent of h. Thus (2.5) holds with $\gamma_0 = C^{-2}$.

Now we consider (2.6). Let $\tilde{S}_h \equiv \{v \in S_h : \int_\Omega v\,dx = 0\}$.

Lemma 5. There is a constant $C_1 > 0$, independent of h, such that

$$\inf_{\varphi\in S_h} \sup_{v\in \tilde{S}_h} \frac{|\int_\Omega \nabla v \cdot \nabla\varphi\,dx|}{\|v\|_{0,h}\|\varphi\|_{2,h}} \geq C_1 \qquad \forall\ h .$$

Proof. We first note that

$$(4.7) \qquad \inf_{\varphi\in\tilde S_h} \sup_{v\in\tilde S_h} \frac{\left|\int_\Omega \nabla v \cdot \nabla\varphi \, dx\right|}{\|v\|_{0,h}\|\varphi\|_{2,h}} = \inf_{v\in\tilde S_h} \sup_{\varphi\in\tilde S_h} \frac{\left|\int_\Omega \nabla v \cdot \nabla\varphi \, dx\right|}{\|v\|_{0,h}\|\varphi\|_{2,h}} .$$

This is a consequence of the fact that an operator and its adjoint have equal norms.

Given $v \in \tilde S_h$ we choose φ to satisfy

$$\begin{cases} \varphi \in \tilde S_h \\ \int_\Omega \nabla\varphi \cdot \nabla\xi \, dx = \int_\Omega v\,\xi\, dx \qquad \forall\ \xi \in \tilde S_h . \end{cases}$$

Letting $\xi = v$ and using Lemma 1 we obtain

$$(4.8) \qquad \int_\Omega \nabla v \cdot \nabla\varphi \, dx = \int_\Omega v^2 dx \geq C_2 \|v\|_{0,h}^2$$

where $C_2 > 0$ is independent of h.

Now let $\bar\varphi$ be defined by

$$\begin{cases} \bar\varphi \in \tilde H^1(\Omega) \equiv \{u \in H^1(\Omega) : \int_\Omega u\, dx = 0\} \\ \int_\Omega \nabla\bar\varphi \cdot \nabla\xi \, dx = \int_\Omega v\,\xi\, dx \quad \forall\ \xi \in \tilde H^1(\Omega) . \end{cases}$$

Then $\frac{\partial\bar\varphi}{\partial\nu} = 0$ on Γ and, since Ω is convex,

$$(4.9) \qquad \|\bar\varphi\|_2 \leq C\|v\|_0 .$$

φ is the Neumann projection of $\bar\varphi$ into $\tilde S_h$ and it is well-known that

$$(4.10) \qquad \|\varphi - \bar\varphi\|_1 \leq C\,h\|\bar\varphi\|_2 .$$

Let $\bar{\bar\varphi}$ be the piecewise linear interpolant of $\bar\varphi$.

Since $\bar\varphi \in H^2(\Omega)$ and $\frac{\partial\bar\varphi}{\partial\nu} = 0$ on Γ we see from the definition of $\|\cdot\|_{2,h}$ and from (4.9) that

$$(4.11) \qquad \|\bar\varphi\|_{2,h} = \|\bar\varphi\|_2 \leq C\|v\|_0 .$$

From Lemma 4 with $k = 1$ and $r = 2$, and (4.10) we have

$$(4.12) \qquad \|\bar\varphi - \bar{\bar\varphi}\|_{2,h} \leq C\|\bar\varphi\|_2 \leq C\|v\|_0 .$$

Using Lemma 2, (4.9), (4.10), and standard approximability results we find that

$$(4.13)\quad \|\varphi - \bar{\bar{\varphi}}\|_{2,h} \le C\,h^{-1}\|\varphi - \bar{\bar{\varphi}}\|_1$$

$$\le C\,h^{-1}(\|\varphi - \bar{\varphi}\|_1 + \|\bar{\varphi} - \bar{\bar{\varphi}}\|_1)$$

$$\le C\,h^{-1}(h\|\bar{\varphi}\|_2 + h\|\bar{\varphi}\|_2)$$

$$\le C\|v\|_0 .$$

Now, using (4.11)-(4.13) we have

$$(4.14)\quad \|\varphi\|_{2,h} \le \|\varphi - \bar{\bar{\varphi}}\|_{2,h} + \|\bar{\bar{\varphi}} - \bar{\varphi}\|_{2,h} + \|\bar{\varphi}\|_{2,h}$$

$$\le C_3\,\|v\|_0 \le C_3\,\|v\|_{0,h}$$

where C_3 is independent of h.

Combining (4.8) and (4.14) we get

$$(4.15)\quad \inf_{v\in\tilde{S}_h}\ \sup_{\varphi\in\tilde{S}_h} \frac{\left|\int_\Omega \nabla v\cdot\nabla\varphi\,dx\right|}{\|v\|_{0,h}\ \|\varphi\|_{2,h}} \ge \frac{C_2}{C_3} \equiv C_1 > 0 .$$

The desired result now follows from (4.7) and (4.15).

<u>Theorem 3</u>. There is a constant $k_0 > 0$, independent of h, such that

$$(4.16)\quad \sup_{v\in V_h} \frac{|b_h(v,\varphi)|}{\|v\|_{0,h}} \ge k_0\|\varphi\|_{2,h} \qquad \forall\ \varphi\in W_h \ \text{ and } \forall\, h ,$$

i.e., (2.6) is satisfied.

<u>Proof</u>. Let $\varphi \in W_h$ and set $e = \frac{1}{|\Omega|}\int_\Omega \varphi\,dx$. Then $|e| \le C\|\varphi\|_0$ and $\tilde{\varphi} \equiv \varphi - e \in \tilde{S}_h$. By Lemma 5 there is a $v_1 \in \tilde{S}_h$ such that

$$(4.17)\quad b_h(v_1,\varphi) = b_h(v_1,\tilde{\varphi}) = -\int_\Omega \nabla v_1\cdot\nabla\tilde{\varphi}\,dx \ge \|\tilde{\varphi}\|_{2,h}^2 \ge \|\varphi\|_{2,h}^2 - C_4\|\varphi\|_0^2$$

and

$$(4.18)\quad \|v_1\|_{0,h} \le C\|\tilde{\varphi}\|_{2,h} \le C_5\|\varphi\|_{2,h} .$$

We also know that

$$(4.19)\quad -b_h(\varphi,\varphi) = \int_\Omega |\nabla\varphi|^2 dx \ge C_6\|\varphi\|_0^2$$

and

$$(4.20)\quad \|\varphi\|_{0,h} \le C_7\,\|\varphi\|_{2,h} .$$

Now let $v = v_1 - C_4 C_6^{-1} \varphi$. Then, using (4.17)-(4.20), we have

(4.21) $$b_h(v,\varphi) \geq \|\varphi\|^2_{2,h}$$

and

(4.22) $$\|v\|_{0,h} \leq (C_5 + C_4\, C_7\, C_6^{-1})\, \|\varphi\|_{2,h} .$$

Combining (4.21) and (4.22) we have (4.16) with $C = (C_5 + C_4\, C_7\, C_6^{-1})^{-1}$.

We are now ready to apply Theorem 1 to analyze the Ciarlet-Raviart method. We obtain

$$\|u-u_h\|_{0,h} + \|\psi - \psi_h\|_{2,h} \leq C(\inf_{\chi \in V_h} \|u - \chi\|_{0,h} + \inf_{\eta \in W_h} \|\psi - \eta\|_{2,h}) .$$

Suppose $\psi \in H^r(\Omega)$, $r \geq 3$, and suppose $k \geq 2$. Using Lemmas 3 and 4 we obtain

(4.23) $$\|u - u_h\|_{0,h} + \|\psi - \psi_h\|_{2,h} \leq C\, h^{s-2} \|\psi\|_s$$

where $s = \min(r,k+1)$. From (4.23) we get

(4.24) $$\|u - u_h\|_0 \leq C\, h^{s-2}\, \|\psi\|_s .$$

In addition, (4.23) yields the estimates

(4.25a) $$\left(\int_{\Gamma_h} |u - u_h|^2 ds\right)^{1/2} \leq C\, h^{s-5/2}\, \|\psi\|_s ,$$

(4.25b) $$\left(\sum_{T \in \mathcal{T}_h} \|\psi - \psi_h\|^2_{2,T}\right)^{1/2} \leq C\, h^{s-2}\, \|\psi\|_s ,$$

and

(4.25c) $$\left(\int_{\Gamma_h} \left|J \frac{\partial \psi_h}{\partial \nu}\right|^2 ds\right)^{1/2} \leq C\, h^{s-3/2}\, \|\psi\|_s .$$

We now derive an estimate for $\|\psi - \psi_h\|_1$ by means of the well-known duality argument. Given $d \in H^{-1}(\Omega)$, let θ be the solution of

$$\begin{cases} \Delta^2 \theta = d & \text{in } \Omega \\ \theta = \dfrac{\partial \theta}{\partial \nu} = 0 & \text{on } \Gamma . \end{cases}$$

If we let $w = -\Delta\theta$ then from (4.2) we have

(4.26) $$\|\theta\|_3 + \|w\|_1 \leq C\|d\|_{-1} .$$

Also, from the discussion following equations (4.4) we know that the pair (w,θ) satisfies

$$(d,\varphi)_0 = -a_h(v,w) - b_h(w,\varphi) - b_h(v,\theta) \qquad \forall\ (v,\varphi) \in H_h^0 \times (H_h^2 \cap H_0^1).$$

Setting $v = u - u_h$ and $\varphi = \psi - \psi_h$, using the exact equations (4.4), and the Ritz-Galerkin equations (4.6) we get

$$(d, \psi - \psi_h)_0 = -a_h(u - u_h, w) - b_h(w, \psi - \psi_h) - b_h(u - u_h, \theta)$$
$$= -a_h(u - u_h, w - z) - b_h(w - z, \psi - \psi_h) - b_h(u - u_h, \theta - \mu)$$
$$\forall \quad (z, \mu) \in V_h \times W_h .$$

Thus, using (2.1), (2.2), (4.26), and Lemma 3 and 4 we get

$$|(d, \psi - \psi_h)| \le C(\|u - u_h\|_{0,h} + \|\psi - \psi_h\|_{2,h}) \inf_{z \in V_h} \|w - z\|_{0,h} + \|u - u_h\|_{0,h} \inf_{\mu \in W_h} \|\theta - \mu\|_{2,h}$$

(4.27)

$$\le C\, h\, \|d\|_{-1}(\|u - u_h\|_{0,h} + \|\psi - \psi_h\|_{2,h}).$$

Finally, combining (4.23) and (4.27), we have

(4.28) $$\|\psi - \psi_h\|_1 = \sup_{d \in H^{-1}(\Omega)} \frac{|(d, \psi - \psi_h)|}{\|d\|_{-1}} \le C\, h^{s-1} \|\psi\|_s$$

where $s = \min(r, k+1)$.

Estimates (4.24) and (4.28) improve on those in Ciarlet-Raviart [9]. Scholz [23] obtained (4.24) under the assumption that Γ is smooth. (4.24) and (4.28) were also obtained by Falk-Osborn [12]. Note that the approach of this paper does not yield error estimates for the case $k=1$ for the method studied in this subsection (and also for the method of Subsection b); for this case the reader is referred to Scholz [24]. Using L_∞-estimate techniques Scholz [24] has shown that $\|u - u_h\|_0 = O(h^{s-3/2})$ under different assumptions than those made in (4.24). In [25] it is shown that in any subdomain $\Omega_0 \subset\subset \Omega$, $\|u - u_h\|_{0,\Omega_0}$ is of "nearly" the same order as $\|\psi - \psi_h\|_{0,\Omega}$, provided ψ is sufficiently smooth. Finally we note that our approach allows the treatment of the case when $g \in (H_h^2)' - H^{-1}(\Omega)$. For example, we could treat the case where g is the Dirac function, which corresponds to a concentrated load in plate theory.

Estimates (4.25) are new for this problem. (4.25c) provides an estimate on the rate at which the jumps in the normal derivatives of ψ_h across interelement boundaries is converging to zero and also contains the estimate

$$\int_\Gamma \left|\frac{\partial \psi_h}{\partial \nu}\right|^2 ds \le C\, h^{s-3/2} \|\psi\|_s .$$

b) Herrmann-Miyoshi method

In this subsection we consider another mixed method for the approximate solution of (4.1). In this method the auxiliary variable is the matrix of second order partial derivatives of ψ.

For $T \in \mathcal{T}_h$ and $\underset{\sim}{v} = (v_{ij})$, $1 \le i,j \le 2$, with $v_{ij} \in H^1(T)$ and $v_{12} = v_{21}$ we set

$$M_\nu(\underset{\sim}{v}) = \sum_{i,j=1}^{2} v_{ij}\nu_j\nu_i$$

and

$$M_{\nu\tau}(\underset{\sim}{v}) = \sum_{i,j=1}^{2} v_{ij}\nu_j\tau_i$$

where $\nu = (\nu_1,\nu_2)$ is the unit outward normal and $\tau = (\tau_1,\tau_2) = (\nu_2,-\nu_1)$ is the unit tangent along ∂T. We note that

(4.29) $$\sum_{i,j=1}^{2} \int_T \left(v_{ij} \frac{\partial^2\varphi}{\partial x_i \partial x_j} + \frac{\partial v_{ij}}{\partial x_j}\frac{\partial\varphi}{\partial x_i}\right)dx = \int_{\partial T} \left(M_\nu(\underset{\sim}{v}) \frac{\partial\varphi}{\partial\nu} + M_{\nu\tau}(\underset{\sim}{v}) \frac{\partial\varphi}{\partial\tau}\right)ds$$

for all $\varphi \in H^2(T)$. On

$$\mathring{V}_h(\Omega) \equiv \{\underset{\sim}{v} = (v_{ij}),\ 1 \le i,j \le 2:\ v_{12} = v_{21},\ v_{ij} \in H^1(T)\ \forall\ T \in \mathcal{T}_h\text{, and } M_\nu(\underset{\sim}{v}) \text{ is continuous across interelement boundaries}\}$$

we define

$$\|\underset{\sim}{v}\|_{0,h}^2 = \sum_{i,j} \int_\Omega |v_{ij}|^2 dx + h\int_{\Gamma_h} |M(\underset{\sim}{v})|^2 ds,$$

where, on an interior edge $T' = \partial T^1 \cap \partial T^2$ of $\mathcal{T}_h$, we set $M(\underset{\sim}{v}) = M_{\nu^1}(\underset{\sim}{v}) = M_{\nu^2}(\underset{\sim}{v})$, and on a boundary edge T' of $\mathcal{T}_h$, we set $M(\underset{\sim}{v}) = M_\nu(\underset{\sim}{v})$. Then we define V_h to be the completion of $\mathring{V}_h$ with respect to $\|\underset{\sim}{v}\|_{0,h}$. It is clear that

(4.30) $$\|\underset{\sim}{v}\|_{0,h} \le \left(\sum_{i,j} \|v_{ij}\|_{0,h}^2\right)^{1/2}$$

for all $\underset{\sim}{v} \in \mathbf{H}^1(\Omega) = \{\underset{\sim}{v} = (v_{ij}),\ 1 \le i,j \le 2: v_{12} = v_{21},\ v_{ij} \in H^1(\Omega)\}$. When we use the norm $\|\cdot\|_{0,h}$ it will be clear from the context whether we are applying it to scalar-valued or matrix-valued functions. As in Subsection a we let $W_h = H_h^2 \cap H_0^1$. Then the mixed method studied in this subsection is based on the following formulation of (4.1):

Given $g \in H^{-1}(\Omega)$, find $(u,\psi) \in V_h \times W_h$ satisfying

$$(4.31)\begin{cases} \sum_{i,j=1}^{2} \int_\Omega u_{ij} v_{ij}\, dx + \sum_{i,j=1}^{2} \sum_{T \in \mathcal{T}_h} - \int_T v_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j}\, dx + \int_{\Gamma_h} M(v) J \frac{\partial \psi}{\partial \nu}\, ds = 0 \quad \forall\, v \in V_h \\ \\ \sum_{i,j=1}^{2} \sum_{T \in \mathcal{T}_h} - \int_T u_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\, dx + \int_{\Gamma_h} M(u) J \frac{\partial \varphi}{\partial \nu} ds = -\int_\Omega g \varphi\, dx \quad \forall\, \varphi \in W_h . \end{cases}$$

Using (4.29) we can easily establish the relations between (4.1) and (4.31). If ψ is the solution of (4.1) and $u_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}$, then (u,ψ) is a solution of (4.31), and if (u,ψ) is a solution of (4.31), then ψ is the solution of (4.1) and $u_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j}$.

(4.31) is an example of problem P with V_h and W_h as above,

$$a_h(u,v) = \sum_{i,j} \int_\Omega u_{ij} v_{ij}\, dx ,$$

and

$$b_h(u,\varphi) = \sum_{i,j} \sum_T - \int_T u_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j}\, dx + \int_{T_h} M(u) J \frac{\partial \varphi}{\partial \nu}\, ds .$$

Letting S_h be as defined in (3.1), we consider the approximate problem P_h with

$$V_h = \{v = (v_{ij}) : v_{12} = v_{21},\ v_{ij} \in S_h\}$$

and

$$W_h = S_h \cap H_0^1(\Omega) .$$

With this choice for the forms a_h and b_h and spaces V_h and W_h, problem P_h describes the Herrmann-Miyoshi method [15,16,20]. Note that with this method we obtain direct approximations to ψ and $\frac{\partial^2 \psi}{\partial x_i \partial x_j}$ (the displacement and moments in elasticity problems).

In order to apply Theorem 1 we must check (2.1), (2.2), (2.5), and (2.6). (2.1) and (2.2) are immediate. In light of (4.30), the proof of (2.5) is similar to the proof of (2.5) for the method in Subsection a. Finally we consider (2.6). Let $\varphi \in W_h$ be given. By Theorem 3 we know there is a $v \in S_h$ such that

$$\int_\Omega \nabla v \cdot \nabla \varphi\, dx \geq \|\varphi\|_{2,h}^2$$

and

$$\|v\|_{0,h} \leq C \|\varphi\|_{2,h} .$$

Now let $\underset{\sim}{v} = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$. We immediately have $\underset{\sim}{v} \in W_h$,

$$b_h(\underset{\sim}{V},\varphi) = \sum_{i,j} \int_\Omega \frac{\partial v_{ij}}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx$$

$$= \int_\Omega \nabla v \cdot \nabla \varphi \, dx$$

$$\geq \|\psi\|_{2,h}^2 ,$$

and

$$\|\underset{\sim}{v}\|_{0,h} \leq \left(\sum_{i,j} \|v_{ij}\|_{0,h}^2\right)^{1/2} \leq \sqrt{2}\, C\|\varphi\|_{2,h} .$$

This proves (2.6).

We are now ready to apply Theorem 1 to analyze the Herrmann-Miyoshi method. This application is essentially the same as that in Subsection a. We use the approximability results in Lemma 3, as modified for matrix-valued functions with the aid of (4.30), and in Lemma 4. We will just state the results.

Suppose $\psi \in H^r(\Omega)$, $r \geq 3$. Then

(4.32) $$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_{0,h} + \|\psi - \psi_h\|_{2,h} \leq C\, h^{s-2}\|\psi\|_s$$

and

(4.33) $$\|\psi - \psi_h\|_1 \leq C\, h^{s-1}\, \|\psi\|_s$$

where $s = \min(r,k+1)$. From (4.32) we obtain

(4.34) $$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_0 \leq C\, h^{s-2}\, \|\psi\|_s .$$

Estimates (4.33) and (4.34) improve on those in Brezzi-Raviart [7]. Rannacher [22] recently obtained these estimates for the case $k = 2$. Falk-Osborn [12] also proved these estimates. We further note that (4.32) contains additional information corresponding to the mesh dependent norms (cf. (4.25)).

c) Herrmann-Johnson method

In this subsection we consider a further method for the approximate solution of (4.1) in which, as in the case treated in Subsection b, the auxiliary variable is the matrix of second order partials of ψ. Also as in Subsection b, the method is based on the variational formulation (4.31) (the spaces V_h and W_h and the forms a_h and b_h are the same as in Subsection b).

We now consider the problem P_h with

$$V_h = \{\underset{\sim}{v} \in \overset{\circ}{V}_h : v_{ij}\big|_T \in P_{k-1} \quad \forall \; T \in T_h\}$$

and

$$W_h = S_h \cap H_0^1(\Omega) \; .$$

This choice leads to the Herrmann-Johnson method [15,16,17]. Note that this method differs from the Herrmann-Miyoshi method only in the choice of the finite dimensional space V_h.

This example has certain special features which allow an analysis that is rather different than that employed in the previous two examples. These special features involve the existence of two particular projection operators denoted by π_h and Σ_h. We turn to this now.

π_h is defined as in [7, Section 4]. For $\underset{\sim}{v} = (v_{ij})$ with $v_{ij} \in H^1(T)$ and $v_{12} = v_{21}$ we define $\pi_T \underset{\sim}{v} = (w_{ij})$ with $w_{ij} \in P_{k-1}$ and $w_{12} = w_{21}$ by

$$(4.35) \quad \begin{cases} \int_{T'} M_\nu (\underset{\sim}{v} - \pi_T \underset{\sim}{v}) f \, ds = 0 \quad \forall \; f \in P_{k-1} \text{ and for each side } T' \text{ of } T, \\ \\ \int_T [v_{ij} - (\pi_T \underset{\sim}{v})_{ij}] f \, dx = 0 \quad \forall \quad f \in P_{k-2} \; . \end{cases}$$

By Lemma 3 in [6], $\Pi_T \underset{\sim}{v}$ is uniquely determined by (4.35). Now for $\underset{\sim}{v} \in \overset{\circ}{V}_h$ we define $\pi_h \underset{\sim}{v} \in V_h$ by

$$(\pi_h \underset{\sim}{v})\big|_T = \pi_T(\underset{\sim}{v}\big|_T)$$

Since we can write

$$b_h(\underset{\sim}{v},\varphi) = \sum_{T\in\mathcal{T}_h} \{-\sum_{i,j} \int_T v_{ij} \frac{\partial^2\varphi}{\partial x_i \partial x_j} dx + \int_{\partial T} M_\nu(\underset{\sim}{v}) \frac{\partial\varphi}{\partial\nu} ds\}$$

it is clear that

$$(4.36) \qquad b_h(\underset{\sim}{v} - \pi_h \underset{\sim}{v},\varphi) = 0 \qquad \forall \ \varphi \in W_h .$$

Concerning the approximation of $\underset{\sim}{v}$ by $\pi_h \underset{\sim}{v}$ we have

__Lemma 6__. Suppose $\underset{\sim}{v} \in [H^{r-2}(\Omega)]^4 \cap \mathring{V}_h$, $r \geq 3$. Then

$$(4.37) \qquad \|\pi_h \underset{\sim}{v} - \underset{\sim}{v}\|_{0,h} \leq C h^\ell \|\underset{\sim}{v}\|_\ell$$

for $1 \leq \ell \leq \min(k,r-2)$.

__Proof__. In Lemma 4 of [7] it is shown that

$$\|\pi_h \underset{\sim}{v} - \underset{\sim}{v}\|_0 \leq C h^\ell \|\underset{\sim}{v}\|_\ell .$$

Thus it remains to show that

$$(h \int_{\Gamma_h} |M(\pi_h \underset{\sim}{v} - \underset{\sim}{v})|^2 ds)^{1/2} \leq C h^\ell \|\underset{\sim}{v}\|_\ell .$$

Let $T \in \mathcal{T}_h$ and assume T is the image of $\hat{T}$ under the mapping $F(\hat{x}) = B\hat{x} + b$. Given a matrix valued function $\underset{\sim}{w}(x)$ on T we set $\hat{\underset{\sim}{w}}(\hat{x}) = C \underset{\sim}{w} (F(\hat{x})) C^t$, $\hat{x} \in \hat{T}$, where $C = B^{-1}$. (Note that the correspondence between (matrix valued) functions on T and on $\hat{T}$ is different than the one introduced in Section 3.) Recall that $\nu = C^t\hat{\nu}|B^t\nu|$ ((3.5)). Then we have

$$\begin{aligned}
\int_{\partial T} |M_\nu(\underset{\sim}{v} - \pi_T\underset{\sim}{v})|^2 ds &= \textstyle\sum_i \int_{T_i'} |M_\nu(\underset{\sim}{v} - \pi_T\underset{\sim}{v})|^2 ds \\
&= \textstyle\sum_i \int_{T_i'} |\nu^t(\underset{\sim}{v} - \pi_T\underset{\sim}{v})\nu|^2 ds \\
&= \textstyle\sum_i \int_{T_i'} |\hat{\nu}^t C\, B(\hat{\underset{\sim}{v}} - \pi_{\hat{T}}\hat{\underset{\sim}{v}})(F^{-1}(x))B^t C^t \hat{\nu}|^2 |B^t\nu|^4 ds \\
&\leq \|B\|^4 \max|T_i'| \int_{\partial\hat{T}} |M_{\hat{\nu}}(\hat{\underset{\sim}{v}} - \pi_{\hat{T}}\hat{\underset{\sim}{v}})|^2 d\hat{s} \\
&\leq C(\hat{T}) h_T \|B\|^4 |\hat{\underset{\sim}{v}}|^2_{\ell,T}
\end{aligned}$$

$$\leq C(\hat{T}) h_T \|B\|^{2(\ell+2)} \|C\|^4 |\det B|^{-1} |v|^2_{\ell,T}$$

$$\leq \frac{C(\hat{T}) h_{\hat{T}}^4 |\hat{T}|4\sigma^6}{\rho_{\hat{T}}^{2(\ell+2)} \pi} h^{2\ell-1} |v|^2_{\ell,T} .$$

Hence

$$h \int_{\Gamma_h} |M(v - \pi_h v)|^2 ds \leq \sum_T h \int_{\partial T} |M_\nu(v - \pi_T v)|^2 ds$$

$$\leq C h^{2\ell} |v|^2_{\ell,\Omega} .$$

This completes the proof.

The second projection operator Σ_h is the interpolation operator I_h introduced in Section 3. As in the proof of Lemma 5 in [7], for $v \in \mathring{V}_h$ and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ we can write

(4.38) $$b_h(v,\varphi) = -\sum_T \sum_{i,j} \int_T \frac{\partial^2 v_{ij}}{\partial x_i \partial x_j} \varphi\, dx + \sum_{T' \in I_h} \int_{T'} A(T',v)\varphi\, ds + \sum_{a \in J_h} B(a,v)\varphi(a)$$

where I_h is the set of all sides of the triangulation $\mathcal{T}_h$, J_h is the set of all vertices of $\mathcal{T}_h$, and $A(T',v)$ is a polynomial of degree less than or equal to $k-2$ in the variable s. Since for $v \in V_h$ we have $\left.\frac{\partial^2 v_{ij}}{\partial x_i \partial x_j}\right|_T \in P_{k-3}$ and $A(T',v) \in P_{k-2}$, it follows from (4.38) that $\Sigma_h \varphi = I_h \varphi$, as defined in Section 3, satisfies

(4.39) $$b_h(v, \Sigma_h \varphi - \varphi) = 0 \quad \forall\ v \in V_h .$$

Now we are ready to derive the error estimates. First we estimate $\|u - u_h\|_0$. Subtracting (2.4a) from (2.3a) we obtain

(4.40) $$a_h(u - u_h, v) + b_h(v, \psi - \psi_h) = 0 \quad \forall\ v \in V_h .$$

Suppose $v \in Z_h \equiv \{w \in V_h : b_h(w,\varphi) = 0 \ \forall\ \varphi \in W_h\}$. Then, from (4.39) we see that $b_h(v,\varphi) = b_h(v,\Sigma_h\varphi) = 0$ for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. Hence from (4.40) we have

(4.41) $$a_h(u - u_h, v) = 0 \quad \forall\ v \in Z_h .$$

Subtracting (2.4b) from (2.3b) and using (4.36) we see that

$$b_h(\pi_h u - u_h, \varphi) = b_h(u - u_h, \varphi) = 0 \quad \forall\ \varphi \in W_h ,$$

i.e., $\pi_h u - u_h \in Z_h$. Thus, recalling (4.41),

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_0^2 = a_h(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{u} - \underset{\sim}{u}_h)$$

$$= a_h(\underset{\sim}{u} - \underset{\sim}{u}_h, (\underset{\sim}{u} - \underset{\sim}{u}_h) - (\pi_h\underset{\sim}{u} - \underset{\sim}{u}_h))$$

$$= a_h(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{u} - \pi_h\underset{\sim}{u})$$

$$\leq \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_0 \|\underset{\sim}{u} - \pi_h\underset{\sim}{u}\|_0$$

and hence

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_0 \leq \|\pi_h\underset{\sim}{u} - \underset{\sim}{u}\|_0 . \tag{4.42}$$

Suppose now that $\psi \in H^r(\Omega)$, $r \geq 3$. Then from (4.42) and Lemma 6 we have

$$\|\underset{\sim}{u} - \underset{\sim}{u}_h\|_0 \leq C h^s \|\psi\|_{s+2} \tag{4.43}$$

where $s = \min(k, r-2)$.

Now we estimate $\psi - \psi_h$. As in Subsection a we can write

$$(d, \psi-\psi_h)_0 = -a_h(\underset{\sim}{u}-\underset{\sim}{u}_h, \underset{\sim}{w}-\underset{\sim}{z}) - b_h(\underset{\sim}{w}-\underset{\sim}{z}, \psi-\psi_h) - b_h(\underset{\sim}{u}-\underset{\sim}{u}_h, \theta-\mu) \quad \forall\ (\underset{\sim}{z}, \mu) \in V_h \times W_h \tag{4.44}$$

where θ is the solution of

$$\begin{cases} \Delta^2\theta = d \in L_2 & \text{on } \Omega \\ \theta = \dfrac{\partial\theta}{\partial\nu} = 0 & \text{on } \Gamma \end{cases}$$

and $w_{ij} = \dfrac{\partial^2\theta}{\partial x_i \partial x_j}$. We note that $(\underset{\sim}{w}, \theta)$ satisfies

$$\begin{cases} a_h(\underset{\sim}{w}, \underset{\sim}{v}) + b_h(\underset{\sim}{v}, \theta) = 0 & \forall \quad \underset{\sim}{v} \in V_h \\ b_h(\underset{\sim}{w}, \varphi) = -\int_\Omega d\,\varphi\,dx & \forall \quad \varphi \in W_h \end{cases} \tag{4.45}$$

(cf. (4.31)). In (4.44) let $\underset{\sim}{z} = \pi_h\underset{\sim}{w}$ and $\mu = \Sigma_h\theta$. This gives

$$(d, \psi-\psi_h)_0 = -a_h(\underset{\sim}{u} - \underset{\sim}{u}_h, \underset{\sim}{w} - \pi_h\underset{\sim}{w}) - b_h(\underset{\sim}{w}-\pi_h\underset{\sim}{w}, \psi-\psi_h) - b_h(\underset{\sim}{u}-\underset{\sim}{u}_h, \theta-\Sigma_h\theta) . \tag{4.46}$$

We now estimate each term in (4.46).

Using (4.36), (4.39), (4.45), and Lemma 3 we have

$$|b_h(\underset{\sim}{w}-\pi_h\underset{\sim}{w}, \psi-\psi_h)| = |b_h(\underset{\sim}{w}-\pi_h\underset{\sim}{w}, \psi-\Sigma_h\psi)| \tag{4.47}$$

$$= |b_n(\underset{\sim}{w}, \psi-\Sigma_h\psi)|$$

$$= |(d, \psi-\Sigma_h\psi)_0|$$

$$\leq \|d\|_0 \|\psi-\Sigma_h\psi\|_0$$

$$\leq C h^s \|\psi\|_s \|d\|_0$$

where $s = \min(r-1, k+1)$.

In our estimate for the third term on the right side of (4.46) we treat the cases $k \geq 2$ and $k = 1$ separately. First assume $k \geq 2$. Then, using (4.26), (4.39), and Lemmas 4 and 6, we find that

$$(4.48) \qquad \begin{aligned} |b_h(\underset{\sim}{u}-\underset{\sim}{u}_h, \theta-\Sigma_h\theta)| &= |b_h(\underset{\sim}{u}-\pi_h\underset{\sim}{u}, \theta-\Sigma_h\theta)| \\ &\leq C\|\underset{\sim}{u}-\pi_h\underset{\sim}{u}\|_{0,h}\|\theta-\Sigma_h\theta\|_{2,h} \\ &\leq C\, h^{s-1}\|\underset{\sim}{u}\|_{s-1}\, h\|\theta\|_3 \\ &\leq C\, h^s\|\psi\|_{s+1}\|d\|_0 \end{aligned}$$

where $s = \min(r-1,k+1)$. Now suppose $k = 1$. Then, using (4.26), (4.31), (4.39), and Lemma 3 we have

$$(4.49) \qquad \begin{aligned} |b_h(\underset{\sim}{u}-\underset{\sim}{u}_h, \theta-\Sigma_h\theta)| &= |b_h(\underset{\sim}{u}, \theta-\Sigma_h\theta)| \\ &= |(\Delta^2\psi, \theta-\Sigma_h\theta)_0| \\ &\leq \|\Delta^2\psi\|_0\|\theta-\Sigma_h\theta\|_0 \\ &\leq C\, h^2\|\Delta^2\psi\|_0\|\theta\|_2 \\ &\leq C\, h^2\|\psi\|_4\|d\|_0 \,. \end{aligned}$$

Finally, using (4.26), (4.42), and Lemma 6 we obtain

$$(4.50) \qquad \begin{aligned} |a_h(\underset{\sim}{u}-\underset{\sim}{u}_h, \underset{\sim}{w}-\pi_h\underset{\sim}{w})| &\leq \|\underset{\sim}{u}-\underset{\sim}{u}_h\|_0\|\underset{\sim}{w}-\pi_h\underset{\sim}{w}\|_0 \\ &\leq C\, h^s\|\psi\|_{s+1}\|d\|_0 \end{aligned}$$

where $s = \min(k+1,r-1)$.

Combining (4.46)-(4.50) we have

$$(4.51) \qquad \begin{aligned} \|\psi - \psi_h\|_0 &= \sup_{d\in L_2} \frac{|(d,\psi-\psi_h)_0|}{\|d\|_0} \\ &\leq C\, h^s\|\psi\|_{s+1}\,, \qquad s = \min(k+1,r-1), \text{ if } k \geq 2 \end{aligned}$$

and

$$(4.52) \qquad \|\psi - \psi_h\|_0 \leq C\, h^2\|\psi\|_4 \,, \quad \text{if } k = 1\,.$$

One can also prove that

$$(4.53) \qquad \|\psi - \psi_h\|_1 \leq C\, h^{s-1}\|\psi\|_s \,, \quad s = \min(r,k+1), \text{ if } k \geq 2$$

and

$$(4.54) \qquad \|\psi - \psi_h\|_1 \leq C\, h\|\psi\|_3 \,, \quad \text{if } k = 1\,.$$

Estimate (4.53) improves on estimates in [7]. Estimates (4.43), (4.51)-(4.53) are given in [7], and (4.43) and (4.51)-(4.54) are proved in [12].

Remarks: 1) As in Subsection b we could have shown that the method studied here is stable with respect to the norm $\| \|_{0,h} + \| \|_{2,h}$, and then obtained error estimates in this norm. This approach would have allowed the treatment of the case when $g \in (H_h^2)' - H^{-1}(\Omega)$ (cf. the next to the last paragraph in Subsection a). However, due to the special nature of this example, more refined estimates can be obtained by the analysis sketched above in the case when sufficient regularity of the solution is assumed. Thus the mesh dependent norms play a less central role in the analysis of this method than in previous methods. They are, however, convenient; their use leads to a very natural setting for the study of this example.

2) The analysis in this subsection was based on the projections π_h and Σ_h and the fact that

$$Z_h \subset Z \equiv \{\underset{\sim}{w} : \underset{\sim}{w} \in V_h , b_h(\underset{\sim}{w},\varphi) = 0 \quad \forall \quad \varphi \in H^2(\Omega) \cap H_0^1(\Omega)\}$$

which follows from the existence of Σ_h. For a general discussion of the projections π_h and Σ_h and the condition $z_h \subset Z$ see Falk and Osborn [12] and Fortin [13].

3) In this subsection the mesh family is not required to be quasi-uniform.

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